International Journal of Theoretical Physics, Vol. 41, No. 5, May 2002 (© 2002)

Differential Groupoids and Their Application to the Theory of Spacetime Singularities

Michael Heller[1,3] and Wiesław Sasin[2]

Received September 7, 2001

The transformation groupoid $\Gamma = \overline{OM} \times G$, where $\overline{OM}$ is the total space of the generalized frame G-bundle over spacetime with a singular boundary, is not a Lie groupoid but a differential groupoid, i.e., a smooth groupoid in the category of structured spaces. We define this concept and use it to investigate spacetimes with various kinds of singularities. Any differential transformation groupoid can be represented by an algebra of operators on a bundle of Hilbert spaces defined on the groupoid fibers. This algebra reflects the structure of a given fiber even if it is a fiber over a singularity. It is also shown that any spacetime with singularities can be regarded as a noncommutative space. Its geometry is done in terms of a noncommutative algebra defined on the corresponding differential transformation groupoid. We focus on the structure of "malicious singularities" such as the ones appearing in the beginning and in the end of the closed Friedman universe.

KEY WORDS: differential groupoid; classical singularities; structured spaces; singular boundaries; spacetime.

1. INTRODUCTION

In the present paper, we continue our research on spacetime singularities by using methods of generalized geometries. In the standard approach, classical spacetime singularities are regarded as ideal points of spacetime or as elements of its "singular boundary," and one attempts to collect information on the existence of singularities and possibly on their nature by approaching them, in a kind of a limiting process, from within the spacetime manifold (Clarke, 1993; Hawking and Ellis, 1973). In our program, we are looking for broader mathematical categories than that of sufficiently smooth manifolds which would allow us to regard even the strongest singularities as "internal elements" of a generalized space. In Heller

[1] Vatican Observatory, Vatican City State.
[2] Faculty of Mathematics and Information Science, Warsaw University of Technology, Plac Politechniki 1, Warsaw, Poland.
[3] To whom correspondence should be addressed at ul. Powstańców Warszawy 13/94, 33-110 Tarnów, Poland; e-mail: mheller@wsd.tarnow.pl.

0020-7748/02/0500-0919/0 © 2002 Plenum Publishing Corporation

and Sasin (1995a,b,c) we have applied to this end the theory of structured spaces, developed previously by us in Heller and Sasin (1994). It has turned out that singularities of milder kinds (the so-called regular and quasi-regular singularities, in the classification of Ellis and Schmidt, 1977) nicely surrender to the methods of structured spaces, but the strongest singularities (such as the ones appearing in the closed Friedman world model and in the Schwarzschild solution), called by us malicious singularities, still create serious problems. For instance, the initial and final singularities of the closed Friedman universe form a single point of Schmidt's *b*-boundary which is not Hausdorff separated from the rest of spacetime (Bosshard, 1976; Johnson, 1977). The structured space methods transparently disclose the mechanism of this behavior, but can hardly supply a remedy.

Schmidt's method of defining the *b*-boundary of spacetime consists in constructing the Cauchy completion $\overline{OM}$ of the frame bundle OM over spacetime M (with the help of a Riemannian metric on OM) and "projecting it down" (by using the action of the structural group G on $\overline{OM}$) to obtain $\bar{M} = \partial_b M \cup M$, where $\partial_b M$ is the *b*-boundary of spacetime M. In Heller and Sasin (1996, 1999) we pushed forward the "desingularization" process by constructing a groupoid $\Gamma = \overline{OM} \times G$ and defining on it a noncommutative algebra $\mathcal{A}$. It has turned out that the fibers of Γ, even the ones over malicious singularities, are isomorphic to the entire group G. By using methods of noncommutative geometry we were able to prove several theorems formulating the conditions under which various types of singularities appear (Heller and Sasin, 1999).

The method of constructing a noncommutative space with the help of a smooth groupoid is well known (Connes, 1994, pp. 99–103); however, to apply it to spaces with singularities we had to go beyond the category of smooth manifolds. In our previous works we did that in a more or less implicit way. One of the goals of the present paper is to do this explicitly and in a rigorous manner. This is achieved in Section 3 where we define the concept of the groupoid *in the category of structured spaces* (we call it *differential groupoid*), and explore some of its properties. To make the paper self-contained we give, in Section 2, a short review of necessary tools from the theory of structured spaces.

Our second goal is to further develop our methods in the study of classical singularities appearing in relativistic cosmology and relativistic astrophysics. Main tools serving this end are differential transformation groupoids and suitable (noncommutative) algebras on them. We prepare these tools in Section 4, and test them on simple toy models in Section 5. In Section 6, these tools are applied to the study of malicious singularities and, finally, in Section 7, we summarize our main results. Some information on a given singularity (even if it is a malicious one) is contained in its differential dimension (a concept defined in the theory of structured spaces) and in the isotropy group of the "singular fiber." This group can be thought of as measuring the "strength" of the singularity: for fibers over malicious singularities it is isomorphic to the entire group G; for fibers over milder singularities it is isomorphic to a certain subgroup of G; and for fibers over non-singular points it is isomorphic to the trivial subgroup of G (consisting of its neutral element). These properties are reflected in the structure of the representation of the corresponding differential groupoid in a Hilbert space, and to a certain extent in a noncommutative algebra defined on this groupoid.

2. STRUCTURED SPACES

Let (M, τ) be a topological space, and $\mathcal{C}$ a sheaf of real continuous functions on (M, τ). The symbol $\mathcal{C}(U)$, $U \in \tau$, denotes the cross section of $\mathcal{C}$ on U.

Definition 2.1. The sheaf $\mathcal{C}$ is said to be a *differential structure* on M if for any open set $U \in \tau$ and any functions $f_1, \ldots, f_n \in \mathcal{C}(U)$ and $\omega \in C^\infty(\mathbf{R}^n)$, the superposition $\omega \circ (f_1, \ldots, f_n)$ belongs to $\mathcal{C}(U)$. The pair $(M, \mathcal{C})$ is called the *structured space*; if topology is important we also write $(M, \tau, \mathcal{C})$.

Example 2.2. Let M be a differential manifold, and $\mathcal{C}$ the sheaf of smooth functions on M. $\mathcal{C}$ is a differential structure on M, and the pair $(M, \mathcal{C})$ is a structured space.

Let us consider a presheaf $\mathcal{D}$ on a topological space (M, τ). For any open set $U \in \tau$, a function $f : U \to \mathbf{R}$ is said to be a *local* $\mathcal{D}$*-function* on M, if for any point $p \in U$ there exists a neighborhood V of p and a cross section $g \in \mathcal{D}(V)$ such that $f \mid U \cap V = g \mid U \cap V$. The presheaf of local $\mathcal{D}$-functions on M is denoted by $\mathcal{D}^+$. In fact, $\mathcal{D}^+$ is a sheaf on (M, τ), and this sheaf is isomorphic with the sheaf associated with the presheaf $\mathcal{D}$.

Let now $(M, \mathcal{C})$ be a structured space, and A a nonempty subset of M. Let us consider the presheaf $\mathcal{C} \mid A$ of restrictions of cross sections of $\mathcal{C}$ to open subsets of A. The differential structure $\mathcal{C}_A := (\mathcal{C} \mid A)^+$ is called a differential structure *induced* from M to A, and the pair $(A, \mathcal{C}_A)$ a *structured subspace* of $(M, \mathcal{C})$.

Let $(M, \mathcal{C})$ be a structured space and $p \in M$ any of its points. By $\mathcal{C}_p$ we shall denote the space of all germs of the sheaf $\mathcal{C}$ at the point p, i.e., the set of equivalence classes of the equivalence relation $\equiv_p$ given by

$$f \equiv_p g \Leftrightarrow f \mid W = g \mid W$$

for $f \in \mathcal{C}(U)$, $g \in \mathcal{C}(V)$, $p \in W \subset U \cap V$, $U, V, W \in \tau$. $\mathcal{C}_p$ is a linear algebra over $\mathbf{R}$.

Definition 2.3. A linear mapping $v : \mathcal{C}_p \to \mathbf{R}$, such that

$$v(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}(p) \cdot v(\mathbf{g}) + \mathbf{g}(p) \cdot v(\mathbf{f})$$

for $\mathbf{f}, \mathbf{g} \in \mathcal{C}_p$, is said to be the *tangent vector* to the structured space (M, C) at p. Let us denote by T_pM the linear space of all tangent vectors to $(M, \mathcal{C})$ at p.

Definition 2.4. By the (*local*) *dimension* of a structured space $(M, \mathcal{C})$ at $p \in M$ we understand dim T_pM.

Example 2.5. Let $(M, \tau, \mathcal{C})$ be a structured space, and ρ an equivalence relation on M. We form the quotient topological space $(M/\rho, \tau/\rho)$. The quotient sheaf $\mathcal{C}/\rho$, given by

$$(\mathcal{C}/\rho)(U) = \left\{ f : U \to \mathbf{R}\colon f \circ \pi_\rho \mid \pi_\rho^{-1}(U) \in \mathcal{C}\left(\pi_\rho^{-1}(U)\right)\right\}$$

for $U \in \tau/\rho$, with $\pi_\rho : M \to M/\rho$ being the canonical projection, is a differential structure on $(M/\rho, \tau/\rho)$, and $(M/\rho, \tau/\rho, \mathcal{C}/\rho)$ is the quotient structured space.

In principle, there is no limitation on the topology τ appearing in the definition of structured spaces. However, if $\tau = \tau_{\mathcal{C}(M)}$ is the weakest topology in which functions belonging to $\mathcal{C}(M)$ are continuous, then the sheaf $\mathcal{C}$ is determined by the algebra of global sections $\mathcal{C}(M)$, and many geometric problems considerably simplify. The spaces $(M, \tau_C, \mathcal{C})$, which now can simply be denoted by (M, C), where $C = \mathcal{C}(M)$, are called *Sikorski differential spaces* or *S-spaces* for short (Sikorski, 1967, 1971, 1972). Any S-space (M, C) can be interpreted as a structured space $(M, \tilde{C})$, where $\tilde{C}$ is a sheaf of the form $\tilde{C}(U) = C \mid U$, for $U \in \tau_C$. $(M, \mathcal{C}(M))$ is evidently an S-space. The topology $\tau_{\mathcal{C}(M)}$ is, in general, weaker than τ. It can be shown that $\tau = \tau_{\mathcal{C}(M)}$ if and only if, for any $U \in \tau$ and any point $p \in U$, there exists a function $\phi \in \mathcal{C}(M)$, called *bump function*, such that $\phi(p) = 1$ and supp $\phi \subset U$ (Heller *et al.*, 1992).

Let (M, C) be an S-space. We say that its differential structure C is *generated* by a subset C_0 of C, written $C = \text{Gen}\, C_0$, if any function $f \in C$ can locally be presented in the form

$$f \mid U = \omega \circ (g_1, \ldots, g_n) \mid U,$$

where $U \in \tau_C$, $\omega \in C^\infty(\mathbf{R}^n)$, $g_1, \ldots, g_n \in C_0$, $n \in \mathbf{N}$.

Lemma 2.6. *Bounded functions on an S-space generate its differential structure.*

Proof: Let (M, C) be an S-space, and C_b the subalgebra of C of bounded functions on M. Let us consider the family B of functions of the form $B = \{\alpha \circ f; f \in C\}$, where $\alpha : \mathbf{R} \to \mathbf{R}$ is given by $\alpha(x) = \tan^{-1} x$. Of course, $B \subset C_b$.

We shall show that $C = \text{Gen}\, B$. Indeed, let f be any function of C, and p any point of M. Let us suppose that the point $\alpha(f(p))$ has a neighborhood V. We define the function $\beta(x) = \tan x \cdot \phi(x)$ if $x \in V$, and $\beta(x) = 0$ otherwise, where ϕ is a bump function centered at $f(p)$. It can be seen that

$$f \mid U = \beta \circ (\alpha \circ f) \mid U,$$

where $U = (\alpha \circ f)^{-1}(V_0)$ with V_0 a neighborhood of $f(p)$ contained in V on which the bump function $\phi \equiv 1$. This means that the differential structure C is generated by B and consequently also by C_b. $\square$

Definition 2.7. Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be structured spaces. A continuous mapping $f : M \to N$ is called *smooth* if, for any cross section $g \in \mathcal{D}(U)$, the superposition $g \circ (f \mid f^{-1}(U))$ is an element of $\mathcal{C}(f^{-1}(U))$. In such a case, we write $f : (M, \mathcal{C}) \to (N, \mathcal{D})$.

The set of all structured spaces as objects with smooth mappings as morphisms constitutes a category—the *category of structured spaces*. Geometry of structured spaces was systematically developed in Heller and Sasin (1995a).

Structured spaces are a suitable tool for investigating various spaces with singularities (Heller, 1992; Heller and Sasin, 1995a,b,c, 1996, 1999). In all these studies Schmidt's construction of the b-boundary of spacetime (Schmidt, 1971) proved to be useful. Now, we prepare a more general setting (in the category of structured spaces) for this construction.

Let $(\bar{M}, \mathcal{C})$ be a structured space such that $\bar{M} = M \cup \partial M$ with M a smooth manifold which is open and dense in $\bar{M}$. The ∂M is called the *singular boundary* of M (in particular, it can be Schmidt's b-boundary of spacetime). Further let $\pi : T\bar{M} \to \bar{M}$ be the tangent bundle over $\bar{M}$ (for details see Heller and Sasin, 1995b), and let us consider the Whitney sum

$$\pi^n = (\pi \times \cdots \times \pi) \mid T^k\bar{M} : T\bar{M} \oplus \cdots \oplus T\bar{M} \to \bar{M}.$$

We define $\overline{OM} := OM \cup (\pi^n)^{-1}(\partial M)$, where OM is the total space of the frame bundle over M. In the "singular fibers" we choose all possible sets of n tangent vectors. In this way, we obtain the *generalized frame bundle* over $\bar{M}$

$$\bar{\pi} : \overline{MO} \to \bar{M},$$

where $\bar{\pi} = \pi^n \mid OM$.

Let G be the structural group of the bundle $(\overline{OM}, \bar{\pi}, \bar{M})$. G acts on $\overline{OM}$ to the right $\overline{OM} \times G \to \overline{OM}$. Let us notice that

$$\overline{OM}/G = OM/G \cup (\pi^n)^{-1}(\partial M)/G = M \cup \partial M.$$

Example 2.8. Let $\bar{M} = M \cup \{x_0\}$ such that the only open neighborhood of x_0 in $\bar{M}$ is the entire $\bar{M}$ (x_0 is a malicious singularity, see below Section 6). In this case,

$$(\pi^n)^{-1}(\{x_0\}) = \{(0, \ldots, 0)\}$$

since dim $T_{x_0}\bar{M} = 0$. Indeed, the global cross sections of the structured sheaf $\mathcal{C}$ are constant functions. This means that the space of its germs is isomorphic with **R**, and consequently each tangent vector at x_0 is the zero vector (Heller and Sasin, 1994).

3. DIFFERENTIAL GROUPOIDS

We begin this section with a brief description of the groupoid structure (see, for instance, Paterson, 1999, Chapter 1) mainly to fix the notation. By a *groupoid* we mean a set Γ with a distinguished subset $\Gamma^2 \subset \Gamma \times \Gamma$ and two mappings, one $\cdot : \Gamma^2 \to \Gamma$ defined by $(x, y) \mapsto x \cdot y$, called *multiplication*, and another $^{-1} : \Gamma \to \Gamma$ defined by $x \mapsto x^{-1}$ such that $(x^{-1})^{-1} = x$, called *inversion* (we treat multiplication in the same way as the composition of functions, i.e., we begin multiplying from the right). Γ^2 is called the *set of composable elements*. Both mappings are supposed to satisfy the following conditions:

(i) if $(x, y), (y, z) \in \Gamma^2$ then $(xy, z), (x, yz) \in \Gamma^2$ and $(xy)z = x(yz)$,
(ii) $(y, y^{-1}) \in \Gamma^2$ for all $y \in \Gamma$, and if $(x, y) \in \Gamma^2$ then $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

We also define the *set of units* $\Gamma^0 = \{xx^{-1}\colon x \in \Gamma\} \subset \Gamma$, and introduce the following mappings: the *source mapping* $d : \Gamma \to \Gamma^0$ by $d(x) = x^{-1}x$, and the *target mapping* $r : \Gamma \to \Gamma^0$ by $r(x) = xx^{-1}$. Let us notice that two elements x, $y \in \Gamma$ can be composed (multiplied) with each other, i.e., $(x, y) \in \Gamma^2$, if and only if $d(x) = r(y)$.

For each $u \in \Gamma^0$ let us define the sets

$$\Gamma_u = \{x \in \Gamma :\ d(x) = u\} = d^{-1}(u)$$

and

$$\Gamma^u = \{x \in \Gamma :\ r(x) = u\} = r^{-1}(u).$$

Both these sets give different fibrations of Γ. The set $\Gamma_u^u := \Gamma^u \cap \Gamma_u$ is closed under multiplication and inversion. It is called the *isotropy group* at u.

If $H \subset \Gamma$ is closed under multiplication and inversion, it defines the *subgroupoid* of Γ with the set of units $H^0 = d(H) = r(H)$ (Landsman, 1998, p. 271).

Groupoids can be regarded as generalizing and unifying the properties of groups and equivalence relations. If we think of a group as describing symmetries on the whole set, the groupoid, being a "group with many units," should be thought of as describing partial symmetries. The groupoid structure on a set tells us, just as any equivalence relation, not only which elements of this set are equivalent, "but it also parametrizes the different ways in which two elements can be equivalent" (Weinstein, 1996). In fact, every equivalence relation is trivially a groupoid.

The above is purely algebraic construction, but we can add to it the smoothness structure. This is done in the following way (Paterson, 1999).

Definition 3.1. A groupoid Γ is a *smooth* or *Lie groupoid* if

(i) Γ is a smooth manifold,
(ii) Γ^0 is a Hausdorff submanifold of Γ,
(iii) every Γ^u and Γ_u is Hausdorff in the relative topology,
(iv) the multiplication and inversion maps are smooth,
(v) the range and source maps are submersions.

In the following definition we generalize the above concept to the category of structured spaces.

Definition 3.2. A groupoid Γ is a *differential groupoid* if

(i) Γ is a structured space,
(ii) $\Gamma^0, \Gamma^u, \Gamma_u$, for every $u \in \Gamma^0$, are Hausdorff structured spaces,
(iii) multiplication and inversion maps are smooth
(iv) the range and source mappings are submersions (called also coregular mappings).

In this definition, smoothness should be understood in the sense of Definition 2.7, and submersion in the category of structured spaces is defined in the following way.

Definition 3.3. A smooth mapping

$$f : (M, \mathcal{C}) \to (N, \mathcal{D})$$

is said to be a *submersion* or a *coregular mapping* if for every $x \in M$ there exist an open neighborhood U of x, an open neighborhood V of $f(x)$, a structured space $(N_0, \mathcal{D}_0)$, and a diffeomorphism

$$\phi : (U, \mathcal{C}_U) \to (V, \mathcal{D}_V) \times (N_0, \mathcal{D}_0)$$

such that $pr_1 \circ \phi = f \mid U$, where $pr_1 : V \times N_0 \to V$ is the obvious projection.

This definition is a straightforward generalization of the definition given by Waliszewski (1972, 1975) for S-spaces. Roughly speaking, it says that a mapping is a submersion if it is locally the projection of a Cartesian product.

Lemma 3.4. *A subgroupoid H of a differential groupoid Γ is a differential groupoid.*

Proof: H is closed with respect to multiplication and inversion by the subgroupoid definition. H is also a structured subspace of Γ with the differential structure induced from that of Γ, and multiplication and inversion in H are smooth as restrictions of smooth mappings. □

In many applications it is important to have a Haar system on groupoids. It is evident that if we restrict a differential groupoid to its regular fibers, i.e., to the fibers over a smooth manifold, we obtain the Lie subgroupoid, and the problem of the existence of Haar systems on it reduces to the problem of the existence of Haar systems on Lie groupoids (see, for instance, Paterson, 1999, pp. 61–64). In the cases considered in the following sections, a Haar measure on the corresponding differential groupoids always exists (see below).

4. DIFFERENTIAL TRANSFORMATION GROUPOID AND ITS C^*-ALGEBRA

Transformation groupoids (also called action groupoids) form an important class of Lie groupoids (see, for instance, Landsman, 1998). In this section, we present their generalization to the category of structured spaces.

Let E be a structured space, and let a group G act on it to the right, $E \times G \to E$. This action leads to the bundle $(E, \pi_M, M = E/G)$. The Cartesian product $\Gamma = E \times G$ has the structure of a groupoid (we call it a *differential transformation groupoid*). The elements of Γ are pairs $\gamma = (p, g)$, where $p \in E$ and $g \in G$. Two such pairs $\gamma_1 = (p, g)$ and $\gamma_2 = (pg, h)$ are composed in the following way:

$$\gamma_2\gamma_1 = (pg, h)(p, g) = (p, gh),$$

and the inverse of (p, g) is (pg, g^{-1}). If we represent $\gamma = (p, g)$ as an arrow beginning at p and ending at pg, then two arrows γ_1 and γ_2 can be composed if the beginning of γ_2 coincides with the end of γ_1. The set of units is

$$\Gamma^0 = \{\gamma^{-1}\gamma : \gamma \in \Gamma\} = \{(p, e) : p \in E\}.$$

To see that this groupoid is indeed a differential groupoid (in the category of structured spaces), it is enough to show that mappings d and r are smooth surjections (in the category of differential spaces). But it is clear from the fact that the mappings $\phi : \Gamma \to \Gamma^0 \times G$ and $\psi : \Gamma \to \Gamma^0 \times G$, given by

$$\phi(p, h) = ((p, e), h), \quad \text{for } (p, h) \in \Gamma,$$

and

$$\psi(p, h) = ((ph, e), h), \quad \text{for } (p, h) \in \Gamma,$$

are diffeomorphisms satisfying $pr_1 \circ \phi = d$ and $pr_1 \circ \psi = r$.

Let us also notice that $\Gamma = E \times G$ is a Hausdorff groupoid. Indeed, the Cartesian product $E \times G$ (G can even be a discrete group) of two Hausdorff spaces is a Hausdorff space.

We have

$$\Gamma_{(p,e)} = \{(p, g) \colon g \in G\}$$

and

$$\Gamma^{(p,e)} = \{(ph^{-1}, h) \colon h \in G\}.$$

In what follows, we shall abbreviate the symbols $\Gamma_{(p,e)}$ and $\Gamma^{(p,e)}$ to Γ_p and Γ^p, respectively. If an element $\gamma = (p, g) \in \Gamma$ is visualized as an arrow from p to pg, the set Γ_p can be thought of as the set of arrows which begin in (p, e), and the set Γ^p as the set of arrows which end at (p, e).

Now, we prove the useful lemma.

Lemma 4.1. *$\bigcup_{g \in G} \Gamma_{pg} = \bigcup_{g \in G} \Gamma^{pg} = pr^{-1}(x)$, where $pr = \pi_M \circ \pi_E$ and $x = \pi_M(p)$.*

The lemma says that the counterimage of $x \in M$ under the projection pr is a sum of g-equivalent groupoid fibers of arrows that begin at $p \in E$, resp. end at p, where p is such that $\pi_M(p) = x$. Two fibers Γ_p and Γ_q (Γ^p and Γ^q) are defined to be equivalent if there exists $g \in G$ such that $q = pg$.

Proof: We prove the second part of the lemma; the first one goes analogously. Let $\gamma \in \bigcup_{g \in G} \Gamma^{(pg, g^{-1})}$. This implies that $\gamma = (pgh^{-1}, h)$, and

$$pr(\gamma) = \pi_M(\pi_E(pgh^{-1}, h)) = \pi_M(pgh^{-1}) = \pi_M(p) = x,$$

i.e., $\bigcup_{g \in G} \Gamma^{(pg, g^{-1})} \subset pr^{-1}(x)$.

Now, let $\gamma \in pr^{-1}(x)$. This implies that if $\gamma = (q, h)$, say, then

$$pr_M(pr_E(q, h)) = \pi_M(q) = x \Rightarrow \exists g \in G, q = pg.$$

Therefore, γ is of the form (pg, h). Hence, $r(\gamma) = (pgh, e)$, and this means that $\gamma \in \bigcup_{g \in G} \Gamma^{(pg,e)}$. □

In all cases considered below (unless the contrary statement is evident from the context), we shall assume that on the group G there exists a left Haar measure. Since all fibers of the groupoid Γ are isomorphic with G, the Haar system can be defined on Γ, and Γ can be treated as a locally compact Hausdorff groupoid (see Paterson, 1999, p. 32).

We shall use the following representation of the groupoid Γ. Every element γ of the groupoid Γ is represented as an operator from the Hilbert space $\mathcal{H}^{d(\gamma)}$ to the Hilbert space $\mathcal{H}^{r(\gamma)}$ where $\mathcal{H}^u = L^2(\Gamma^u)$. We thus define the representation

$$\mathcal{L} : \Gamma \to \mathrm{End}\big(\mathcal{H}^{d(\gamma)}, \mathcal{H}^{r(\gamma)}\big)$$

by

$$\mathcal{L}(\gamma)(\xi)(\eta) = \xi(\gamma^{-1}\eta), \tag{1}$$

where $\xi \in L^2(\Gamma^{d(\gamma)})$, $\eta \in \Gamma^{r(\gamma)}$ (for details see Paterson, 1999, p. 93).

On every groupoid Γ we can define an involutive algebra $\mathcal{A}_c = C_c^\infty(\Gamma, \mathbf{C})$ of compactly supported, complex valued functions. Let $a, b \in \mathcal{A}_c$ and $\gamma \in \Gamma$. Multiplication in $\mathcal{A}_c$ is defined in the following way:

$$(a * b)(\gamma) = \int_{\Gamma_{d(\gamma)}} a(\gamma_2)b(\gamma_1) = \int_{\Gamma_{d(\gamma)}} a\left(\gamma\gamma_1^{-1}\right)b(\gamma_1)\, d\gamma_1, \tag{2}$$

where $\gamma = \gamma_2 \circ \gamma_1$ and the integration is with respect to a Haar measure. The involution is defined by

$$a^*(\gamma) = \overline{a(\gamma^{-1})}.$$

If the group G is noncompact, the algebra $\mathcal{A}_c$ is not unital. In this case, to avoid difficulties connected with this fact, we perform the following construction.

We define the algebra $\mathcal{A}_{\text{proj}} := pr_M^*(C^\infty(M))$. This is of course the commutative algebra with the standard pointwise multiplication of functions (notice that the functions of $\mathcal{A}_{\text{proj}}$ are constant on the equivalence classes of the groupoid fibers, see Lemma 4.1). The idea is to consider the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{\text{proj}}$ which would allow us to recover "in the limit" the geometry of spacetime $M = E/G$ in the case when G is the Lorentz group. In fact, on the strength of Lemma 2.6, to this end it is enough to consider, instead of $\mathcal{A}_{\text{proj}}$, its subalgebra $\mathcal{A}_{\text{proj}}^b$ of bounded functions.

We now define the bilateral action of the algebra $\mathcal{A}_{\text{proj}}^b$ on the algebra $\mathcal{A}_c$ in the natural way

$$(a, f) \to a \cdot f, \qquad (f, a) \to f \cdot a$$

for $a \in \mathcal{A}$, $f \in \mathcal{A}_{\text{proj}}^b$. We evidently have $a \cdot f = f \cdot a$. Now, we define the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{\text{proj}}^b$ with the following operations:

$$(a_1, f_1) + (a_2, f_2) = (a_1 + a_2, f_1 + f_2),$$
$$(a_1, f_1) * (a_2, f_2) = (a_1 * a_2 + f_1 a_2 + f_2 a_1, f_1 f_2),$$
$$(a, f)^* = (a^*, \bar{f}).$$

We shall also use the additive notation by writing $a + f$ instead of (a, f). In this way, we obtain the involutive algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{\text{proj}}^b$ with unit $\mathbf{1} = 0 + \mathbf{1}$, where $\mathbf{1}$ is a constant function having everywhere the value 1. This is the unitization of the algebra $\mathcal{A}_c$ which is essentially equivalent to the compactification of the corresponding noncommutative space (Gracia-Bondía *et al.*, 2001, pp. 13–14).

Let us now define the representation $\pi_q : \mathcal{A} \to \operatorname{End} \mathcal{H}_q$ of the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}^b_{\text{proj}}$ in the Hilbert space $\mathcal{H}_q = L^2(\Gamma_{d(\gamma)}) = L^2(\Gamma_q)$ by

$$\pi_q(a + f)(\xi) = \overset{1}{\pi}_q(a)(\xi) + \overset{2}{\pi}_q(f)(\xi), \tag{3}$$

where

$$\overset{1}{\pi}_q(a)(\xi) = \int_{\Gamma_q} a\left(\gamma\gamma_1^{-1}\right)\xi(\gamma_1)\, d\gamma_1$$

and

$$\overset{2}{\pi}_q(f)(\xi) = f(q, e)(\xi).$$

We also define the "integrated" representation of $\mathcal{A}$, $\pi = \bigoplus_{q\in\Gamma^0} \pi_q$, which is, in fact, a one-parameter family of representations.

Now, we define the norm in the algebra $\mathcal{A}_c$ in the following way:

$$\|a\| = \sup_{q\in\Gamma^0} \left\|\overset{1}{\pi}_q(a)\right\|,$$

and in the algebra $\mathcal{A}^b_{\text{proj}}$

$$\|f\| = \sup_{q\in\Gamma^0} \left\|\overset{2}{\pi}_q(f)\right\|;$$

and finally for the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}^b_{\text{proj}}$

$$\|(a, f)\| = \max\{\|a\|, \|f\|\}.$$

The algebra $\mathcal{A}$ completed with respect to this norm is a C^*-algebra. We could use the algebra $\mathcal{A}$ (or its "operator version" $\pi_p(\mathcal{A})$) to construct a noncommutative space. The geometry of this space would be "fully desingularized" (Heller and Sasin, 1999, Section 7).

5. TOY MODELS

In this section we compute two simple models to illustrate the above construction. Although they are extremely naive, they nicely show subtleties of the "desingularization" process.

5.1. Model $\mathbf{R} \times \mathbf{Z}_2$

In this example, $E = \mathbf{R}$ and $G = \mathbf{Z}_2$, and the action of $\mathbf{Z}_2$ on E is given by

$$(p, 1) \mapsto p \quad \text{and} \quad (p, -1) \mapsto -p.$$

The base space ("spacetime") $M = E/\mathbf{Z}_2$ is clearly a half-line with a singularity at $x = 0$. It is a regular singularity since it can be removed by embedding M in a larger space (in this case, in $\mathbf{R}$). We can think of $p \in E$ as of a frame (a vector) at $x \in M$, which is either unmoved or reflected by $\mathbf{Z}_2$. Then E is the total space of the $\mathbf{Z}_2$-frame bundle over M, and the differential groupoid $\Gamma = E \times \mathbf{Z}_2$ is formed by two disconnected lines. The fibers of the groupoid are

$$\Gamma_p = \{(p, 1), (p, -1)\}, \qquad \Gamma^p = \{(p, 1), (-p, -1)\}.$$

We define the algebra $\mathcal{A}$ on the groupoid $\mathcal{A} = C^\infty(E \times \mathbf{Z}_2, \mathbf{C})$. In this simple model, the usual function multiplication would be all right; however, since we want to illustrate a more general situation, we define the following noncommutative multiplication:

$$(a * b)(\gamma) = \sum_{\gamma = \gamma_2 \circ \gamma_1} a(\gamma_2) b(\gamma_1).$$

Because of the $\mathbf{Z}_2$ grading we can write

$$\mathcal{A} = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : \; a \in C^\infty(\mathbf{R}, \mathbf{C}) \right\}.$$

The subalgebra $\mathcal{A}_{\text{proj}}$ is

$$\mathcal{A}_{\text{proj}} = \left\{ \begin{pmatrix} a \\ a \end{pmatrix} : \; a \in C^\infty(\mathbf{R}, \mathbf{C}), a \text{ is an even function} \right\}.$$

It is surely the differential structure on the half-line.

The representation $\mathcal{L}(\gamma) : E \times \mathbf{Z}_2 \to \text{End}(l^2(\Gamma^{d(\gamma)}, \Gamma^{r(\gamma)}))$ of the groupoid $E \times \mathbf{Z}_2$ can be readily computed. We obviously have

$$\mathcal{L}(p, 1) = \text{id}_{l^2(\Gamma^p)};$$

and since for $\gamma = (p, -1)$ the groupoid fibers are

$$\Gamma^{d(\gamma)} = \{(p, 1), (-p, -1)\}, \qquad \Gamma^{r(\gamma)} = \{(p, -1), (-p, 1)\},$$

we easily compute that $\mathcal{L}$ transforms the basis vectors in $l^2(\Gamma^{d(\gamma)})$ into the basis vectors in $l^2(\Gamma^{r(\gamma)})$ in the following way: $\binom{1}{0} \mapsto \binom{0}{1}$, $\binom{0}{1} \mapsto \binom{1}{0}$. Therefore,

$$\mathcal{L}(p, -1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(For instance, $\mathcal{L}(p, -1)\binom{1}{0}(p, -1) = \binom{1}{0}((p, -1)^{-1}(p, -1)) = \binom{1}{0}(p, 1) = 1.$)

Now, we shall find the representation of the algebra $\mathcal{A}$ in the Hilbert space $l^2(\Gamma_p)$. It can be easily seen that

$$l^2(\Gamma_p) = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} : \; \xi_1, \xi_2 \in \mathbf{C} \right\} \simeq \mathbf{C}^2,$$

and the representation is given by

$$\pi_p(a)(\xi)(\gamma) = \sum_{\gamma=\gamma_2\circ\gamma_1} a(\gamma_2)\xi(\gamma_1).$$

If we choose the basis $(\binom{1}{0}, \binom{0}{1})$ in $\mathbf{C}^2$, we easily compute

$$(\pi_p(a))_{p\in E} = \begin{pmatrix} a(p,+1) & a(-p,-1) \\ a(p,-1) & a(-p,+1) \end{pmatrix}_{p\in E}.$$

(For instance, $\pi_p(a)\binom{1}{0}(p,1) = a(p,1)\binom{1}{0}(p,1) + a(-p,-1)\binom{1}{0}(p,-1) = a(p,1)\cdot 1 + a(-p,-1)\cdot 0 = a(p,1)$.)

Let us now analyze the situation at the singularity $x = 0$. The "singular fibers" are

$$\Gamma_{p_0} = \Gamma^{p_0} = \{(0,1),(0,-1)\}, \qquad \pi_M(p_0) = 0,$$

and $\Gamma_{p_0}^{p_0} = \Gamma_{p_0} \cap \Gamma^{p_0} \simeq \mathbf{Z}_2$ (for nonsingular fibers we evidently have $\Gamma_p^p \simeq \{e\}$).

Since the fibers $\Gamma^{d(\gamma)}$ and $\Gamma^{r(\gamma)}$ over the singularity coincide, the groupoid representation $\mathcal{L}$ reads

$$\mathcal{L}(0,1) = \mathcal{L}(0,-1) = \mathrm{id}_{l^2(\Gamma^0)}.$$

The representation of the algebra $\mathcal{A}$ at the "singular fiber" is clearly of the form

$$\pi_{p_0}(a) = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix},$$

whereas at all other fibers

$$\pi_p(a) = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}.$$

As we can see, in this case, both the representation of the groupoid and the representation of the algebra distinguish the "singular fiber" from other fibers.

5.2. Model $\mathbf{R}\times\mathbf{R}$

In this model, we assume $E = \mathbf{R}$ and $G = (\mathbf{R}, +) = \{T_a\colon\ a \in \mathbf{R}, T_a(x) = x + a\}$. We evidently have the action $E \times G \to E$ given by $(x, a) \mapsto x + a$, which allows us to define the groupoid $\Gamma = \mathbf{R} \times G$. Its fibers are

$$\Gamma_p = \{(p,\tau)\colon \tau \in \mathbf{R}\} = \{p\} \times \mathbf{R},$$

which are, of course, "vertical lines," and

$$\Gamma^p = \{(pt^{-1}, t)\colon\ t \in \mathbf{R}\},$$

which are lines inclined by an angle of 45° with respect to the previous ones (to see this, introduce the coordinates $x = pt^{-1} = p - t$, $y = t$). Therefore, all fibers are equivalent, and consequently $M = E/G = \{\text{point}\}$. The situation is seemingly not unlike in the closed Friedman universe with the b-boundary where topologically everything reduces to a single point. However, we can easily see that the isotropy group $\Gamma_p^p \simeq \{e\}$, for every $p \in E$, which is the typical situation for nonsingular points (for malicious singularities the isotropy group is isomorphic to the entire group G). Therefore, in this example, M is a one-point space and not a malicious singularity.

The representation (1) of the groupoid Γ is

$$\mathcal{L}(p, t)(\xi)(pt, \tau) = \xi(p + t, \tau - t).$$

Also this representation does not show any singularity.

We define the algebra $\mathcal{A} = C_c^\infty(\Gamma, \mathbf{C})$ with the multiplication

$$(a * b)(\gamma) = \int_{\Gamma_p = \mathbf{R}} a(p\tau, t - \tau) b(p, \tau)$$

where we have assumed $\gamma = (p, t)$, $\gamma_1 = (p, \tau)$. This leads to the algebra representation

$$\pi_p(a)(\xi)(\gamma) = \int_{\mathbf{R}} a(p\tau, t - \tau)\xi(p, \tau)$$

for all $p \in E$. Thus although "macroscopically" the considered space consists of a single point, from the "quantum point of view" (as represented by operators on a Hilbert space) it is a rich space.

6. MALICIOUS SINGULARITY

6.1. General Case

We now apply the above machinery to investigate the structure of malicious singularities. A singularity is *malicious* if it is a one-element orbit of the action of the group G on the structured space E and if the subalgebra $C_G^\infty(E) \subset \mathcal{A}$ of G-invariant functions on E (i.e., functions that are constant on the orbits of G) is isomorphic to $\mathbf{C}$. Let then $x_0 \in \partial M$ be a malicious singularity. For simplicity we assume that $\partial M = \{x_0\}$. Therefore, we have $M = E/G = \{x_0\} \cup E_0/G$, where $E_0 = E \backslash (\pi_M)^{-1}(x_0)$. We also have $\pi_M(p_0) = x_0$, $p_0 \in E$. The fiber over the malicious singularity is a fixed point of the action of the group G on E, $p_0 G = p_0$. Besides this fiber the action of G is free, i.e., G acts freely on E_0, i.e., $\pi_M : E_0 \to E_0/M$ is a G-bundle.

Now, we construct the differential groupoid $\Gamma = E \times G$. Since in the malicious singularity $x_0 \in \partial_b M$, $p_0 h = p_0$ for every $h \in G$, we have

$$\Gamma_{p_0} = \Gamma^{p_0} = \{(p_0, h) \colon\ h \in G\} = p_0 \times G \simeq G.$$

From this formula it is evident that the fiber over the malicious singularity can be given the structure of the group manifold G (if G is a Lie group). In this way, also malicious singularities can be represented by well-behaved structures. Therefore, we can truly speak about the "desingularization" procedure. The "largeness" of the isotropy group at the singular fiber can be regarded as a measure of the "strength" of the singularity. Indeed, in the extreme cases, if (p_0, h) lies in the fiber over the malicious singularity then $\Gamma_{p_0}^{p_0} = \Gamma_{p_0} \cap \Gamma^{p_0} \simeq G$, and if $u = (p, h)$ lies in the fiber over a nonsingular point of M then $\Gamma_p^p \simeq \{e\}$. Let us also notice that for the fiber over the malicious singularity $\{x_0\}$ we have (see Lemma 4.1)

$$\bigcup_{g \in G} \Gamma_{pg} = \bigcup_{g \in G} \Gamma^{pg} = pr^{-1}(x_0).$$

The differential dimension of $\bar{M}$ at the singularity x_0 is zero.

The representation of the groupoid in the bundle of Hilbert spaces is given by formula (1). For the fiber over the malicious singularity we have $d(\gamma) = r(\gamma)$, which implies $\gamma^{-1}\eta = \eta$, and consequently $\mathcal{L}(\gamma) = \mathrm{id}_{L^2(\Gamma^{d(\gamma)})}$. We thus have

Proposition 6.1. *The representation $\mathcal{L}(\gamma)$ of the transformation groupoid Γ, for γ being an element of the groupoid fiber over the malicious singularity, is reduced to a single identity operator on the Hilbert space $L^2(\Gamma^{d(\gamma)}) = L^2(\Gamma^{r(\gamma)})$.* □

In physically significant cases (for instance, in the Schmidt's b-boundary construction) G is assumed to be the connected component of the Lorentz group. This group is not compact, and consequently the algebra $\mathcal{A}$ is not unital. Therefore, we must perform the unitization of this algebra (see Section 4). Since in the case with the malicious singularity $\mathcal{A}_{\mathrm{proj}} = \mathbf{C}$, the algebra on the groupoid $\Gamma = E \times G$ is $\mathcal{A} = \mathcal{A}_c \times \mathbf{C}$, and its representation (3) assumes the form

$$\begin{aligned}\pi_p(a + c)(\xi) &= \overset{1}{\pi}_p(a)(\xi) + \overset{2}{\pi}_p(c)(\xi) \\ &= \int_{\Gamma_p} a\left(\gamma\gamma_1^{-1}\right)\xi(\gamma_1)\, d(\gamma_1) + H_c(\xi),\end{aligned}$$

where $H_c(\xi) = c \cdot \xi$ for all $p \in E$.

6.2. Two-Dimensional Friedman Universe

The closed Friedman world model is a classical example of a solution to the Einstein field equations with two malicious singularities. In the present section, we analyze its two-dimensional analogue (Bosshard, 1976; Dodson, 1978) with the help of our machinery. We consider the spacetime

$$N = \{(\eta, \chi) : \ \eta \in (0, T), \chi \in S^1\},$$

where $(0, T) \subset \mathbf{R}$. On N there is the metric

$$ds^2 = R^2(\eta)(-d\eta^2 + d\chi^2)$$

such that $R^2(\eta) \to 0$ as $\eta \to 0$. This corresponds to the initial singularity. We can also assume that there exists the final singularity, i.e., that $R^2(\eta) \to 0$ as $\eta \to T$. However, in the present analysis we shall focus only on the initial singularity.

Let $\frac{1}{R(\eta)}\left[\begin{smallmatrix}\partial_\eta\\ \partial_\chi\end{smallmatrix}\right]$ be a frame in N (the factor $\frac{1}{R(\eta)}$ is needed to guarantee the orthogonality of the frame in the above metric). All frames at a given point $(\eta, \chi) \in N$ can be obtained by rotating

$$\frac{1}{R(\eta)}\begin{bmatrix}\cosh\lambda & \sinh\lambda\\ \sinh\lambda & \cosh\lambda\end{bmatrix}\begin{bmatrix}\partial_\eta\\ \partial_\chi\end{bmatrix}.$$

It can be demonstrated that the mapping

$$\lambda \mapsto \begin{bmatrix}\cosh\lambda & \sinh\lambda\\ \sinh\lambda & \cosh\lambda\end{bmatrix} \tag{4}$$

gives the isomorphism of the group $(\mathbf{R}, +)$ with the matrix group

$$G = \left\{\begin{bmatrix}\cosh t & \sinh t\\ \cosh t & \sinh t\end{bmatrix} :\ t \in \mathbf{R}\right\}.$$

The homomorphism (4) is obvious. To show the homomorphism in the reverse direction let us notice that the matrices of G have the form $\left[\begin{smallmatrix}a & b\\ b & a\end{smallmatrix}\right]$ with $a > 1$ and $a^2 - b^2 = 1$. There exists exactly one λ such that $\sinh\lambda = b$. Therefore, $a = \sqrt{1+b^2} = \sqrt{1+\sinh\lambda^2} = \cosh\lambda$ (the minus sign is excluded since $a > 1$). In the following we shall assume that the above matrices are multiplied by the factor $(R(\eta))^{-1}$. We thus can write the total space of the positively oriented component of the orthonormal frame bundle ON over N in the form

$$ON = \{(\eta, \chi, \lambda):\ \eta \in (O, T), \chi \in S^1, \lambda \in \mathbf{R}\}.$$

This frame bundle is globally trivial. One can easily check that $N = ON/\mathbf{R}$.

Since we have the action $ON \times \mathbf{R} \to ON$ defined by $((\eta, \chi, \lambda), t) \to (\eta, \chi, \lambda + t)$, we can construct the transformation groupoid $\Gamma = ON \times \mathbf{R} = \{(\eta, \chi, \lambda, t):\ \eta \in (O, T), \chi \in S^1, \lambda, t \in \mathbf{R}\}$. If $\gamma \in \Gamma$ we shall write $\gamma = (p, t)$, and $d(\gamma) = p = (\eta, \chi, \lambda)$, $r(\gamma) = pt = (\eta, \chi, \lambda + t)$. The groupoid fibers are as usual

$$\Gamma_p = \{(p, t):\ t \in \mathbf{R}\}, \qquad \Gamma^p = \{(pt^{-1}, t):\ t \in \mathbf{R}\}.$$

Two fibers Γ_p and Γ_q are equivalent if $p = (\eta, \chi, \lambda_1)$ and $q = (\eta, \chi, \lambda_2)$. Let us notice that the structure of the equivalence classes of this relation is very similar to that of the model $\mathbf{R} \times \mathbf{R}$ discussed in Section 5.2.

Γ is obviously a Lie groupoid. It is known (Bosshard, 1976; Dodson, 1978) that the fiber (in Schmidt's fiber bundle) over the malicious singularity "at $\eta = 0$" consists of a one point; let us denote it by p_0. The total space of the generalized fiber bundle (see Section 2) is $\bar{E} = E \cup \{p_0\}$, and the groupoid $\bar{\Gamma} = \bar{E} \times G$ is a differential groupoid.

The groupoid representation (1) in this model assumes the form

$$\mathcal{L}(p, t)\xi(pt, \tau) = \xi(p + t, \tau - t).$$

Over the malicious singularity the fibers $\Gamma^{d(\gamma)}$ and $\Gamma^{r(\gamma)}$ coincide, and the "singular fiber" is represented by the identity operator (see Proposition 6.1).

Since the group $G = \mathbf{R}$ is noncompact we must perform the unitization of the algebra $\mathcal{A}_c = C_c^\infty(\bar{\Gamma}, \mathbf{R})$. Therefore, as the algebra on the groupoid $\bar{\Gamma}$ we assume $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{\text{proj}}$, with the operations as defined in Section 3. Let us notice that if $\bar{a} \in \mathcal{A}_c$ then

$$\bar{a} = \begin{cases} a(\eta, \chi, \lambda, t) & \text{if } (\eta, \chi, \lambda, t) \in \Gamma, \\ \alpha(t) & \text{if } (\eta, \chi, \lambda, t) \in \bar{\Gamma} \backslash \Gamma, \end{cases}$$

and

$$\alpha(t) = \lim_{\eta \to 0} a(\eta, \chi, \lambda, t).$$

Representation (2) of this algebra is

$$\pi_p(a + f)(\xi)(\gamma) = \int_{\Gamma_p} a\left(\gamma\gamma_1^{-1}\right)\xi(\gamma_1) + f(p)\xi,$$

$a \in \mathcal{A}_c,\ f \in \mathcal{A}_{\text{proj}}$, which is a nontrivial operator algebra also in the case when $p = p_0$ lies in the fiber over the malicious singularity.

7. RESULTS

In this paper, we have introduced the groupoid concept in the category of structured spaces. In the same way that structured spaces are substantial generalizations of smooth manifolds, differential groupoids are generalizations of smooth or Lie groupoids. This opens a new field of groupoid applications, both in pure mathematics and in theoretical physics, in domains in which one encounters nonsmooth situations. We have applied this new tool to the study of spacetime singularities as they appear in relativistic physics and obtained the following results:

1. To further develop the procedure of "spacetime desingularization" we have constructed the generalized G-frame bundle over spacetime with a singular boundary (in the category of structured spaces), $\bar{\pi} : \overline{OM} \to M \cup \partial M$ (Schmidt's frame bundle over a spacetime with its b-boundary being a special case), and then the transformation groupoid $\Gamma = \overline{OM} \times G$. Γ is not a Lie groupoid but a Hausdorff differential groupoid (in the category of structured spaces).

2. The groupoid fibers Γ_u and Γ^u, for all $u \in \Gamma^0$, are isomorphic to the group G (also over malicious singularities). For all groupoid fibers over nonsingular points the isotropy group $\Gamma_u \cap \Gamma^u = \Gamma_u^u$ is trivial (i.e., isomorphic with {e}). For singular fibers the isotropy group is nontrivial: for fibers over malicious singularities Γ_u^u is isomorphic to the entire group G, and for weaker singularities Γ_u^u is isomorphic to a subgroup of G. In this way, the isotropy group of a given groupoid fiber can be thought of as measuring the "strength" of the corresponding singularity.
3. The transformation groupoid Γ can be represented in a "bundle of Hilbert spaces" suitably defined on the groupoid fibers in such a way that to a given element γ of Γ (an "arrow") there corresponds an operator transforming the Hilbert space defined on the fiber determined by the beginning of γ to the Hilbert space defined on the fiber determined by the end of γ. In this way, the structure of the groupoid fibers (also of the ones over singularities) is reflected in the structure of operators on the bundle of Hilbert spaces.
4. Spacetime with singularities (even the strongest ones) can be regarded as a noncommutative space. Its geometry can be done in terms of a noncommutative algebra $\mathcal{A}$ defined on the differential transformation groupoid $\Gamma = \overline{OM} \times G$. This algebra can be completed to the C^*-algebra and represented in a Hilbert space defined on the groupoid fibers. The corresponding operator algebra on these Hilbert spaces depends on the geometric structure of a given spacetime with singularities.

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